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The Radius of Convergence of a Cardinal Lagrange Spline Series of Odd Degree

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We give a sharp criterion for the convergence of a Lagrangian cardinal spline series for the integer grid in terms of a "radius of convergence."

Let $S_m = S_m(\mathbb{Z})$ denote the linear space of all functions $s \in C^{m-1}(\mathbb{R})$ for which all restrictions

$$s|_{[j-1,j]} = q_j, \qquad j \in \mathbb{Z},$$

are polynomial functions of degree m, at best.

In what follows

$$m = 2r + 1 \in \mathbb{N}$$

is odd. Like Micchelli [3, 4], Schoenberg [7, 8] and others, we call the elements of S_m cardinal spline functions of degree m. It is well known that the interpolation problem

$$S_m \ni s : s(j) = y_j \qquad \text{for} \quad j \in \mathbb{Z}, \tag{1}$$

has a (unique) solution, if the "sequence" $\{y_j\}_{j\in\mathbb{Z}}$, $y_j \in \mathbb{R}$ for $j \in \mathbb{Z}$, is not growing too fast in absolute value as j tends to $\pm\infty$. Micchelli and Schoenberg use power growth conditions, and Greville *et al.* [1] and Schempp [6] deal also with exponentially growing splines, however, based on a B-spline representation.

We are going to derive a precise order condition, which means that we shall calculate a radius of convergence as is usual in the theory of power series.

In [5] we gave an explicit representation for the Lagrangians, defined by

$$l_j \in S_m, \qquad l_j(k) = \delta_{jk} \qquad \text{for} \quad j, k \in \mathbb{Z},$$

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by the aid of the Euler–Frobenius polynomials

$$H_m(t, z) = (1 - z)^{m+1} \, (t + z(\partial/\partial z))^m \, (1/(1 - z)).$$

The work was based essentially on ideas of Meinardus and Merz [2]. Because of $l_j(x) = l_0(x - j)$, we could restrict ourselves on the description of l_0, and because of $l_0(-x) = l_0(x)$ it suffices to represent the restrictions

$$q_j(t) = l_0(j - 1 + t), \qquad 0 \leqslant t \leqslant 1, \quad j \in \mathbb{N}.$$

By [5, formula (13)] we have (with $j - 1$ instead of j)

$$q_j(t) = (1 - t)^m \, \delta_{1,j} - \sum_{k=1}^{r} a_k(t) \, z_k^{\,j} \tag{2}$$

for $j \in \mathbb{N}$ with

$$a_k(t) = H_m(t, z_k)/H_m'(0, z_k), \tag{3}$$

where

$$z_1 < \cdots < z_r < -1 < z_{r+1} < \cdots < z_{2r+1} = 0$$

are the zeros of $H_m(0, z)$.

Note that

$$a_k(t) > 0 \qquad \text{for} \quad 0 < t < 1, \quad k = 1, \ldots, r, \tag{4}$$

is valid.

We try to obtain a solution of the problem (1) in the form

$$s(x) = \sum_{j=-\infty}^{+\infty} y_j l_j(x) \tag{5}$$

for $x \in \mathbb{R}$. The series shall be called "convergent" if both partial sums

$$\sum_{j=-\infty}^{0} \qquad \text{and} \qquad \sum_{j=1}^{+\infty}$$

converge by their own.

To begin with, let $x \in (0, 1)$. Then we have

$$l_j(x) = l_0(x - j) = l_0(j - x) = q_j(1 - x) \qquad \text{for} \quad j \in \mathbb{N},$$
$$l_j(x) = l_0(x + |j|) = q_{|j|+1}(x) \qquad \text{for} \quad -j \in \mathbb{N}_0.$$

Hence we have to investigate the convergence of

$$\sum_{j=-\infty}^{+\infty} y_j l_j(x) = \sum_{j=-\infty}^{0} y_j q_{|j|+1}(x) + \sum_{j=1}^{+\infty} y_j q_j(1-x)$$
$$= \sum_{j=1}^{+\infty} y_{1-j} q_j(x) + \sum_{j=1}^{+\infty} y_j q_j(1-x). \tag{6}$$

By the use of (2) we obtain

$$\sum_{j=1}^{+\infty} y_{1-j} q_j(x) = y_0(1-x)^m - \sum_{j=1}^{+\infty} y_{1-j} \sum_{k=1}^{r} a_k(x)\, z_k^{-j},$$
$$\sum_{j=1}^{+\infty} y_j q_j(1-x) = y_1 x^m - \sum_{j=1}^{+\infty} y_j \sum_{k=1}^{r} a_k(1-x)\, z_k^{-j}.$$

The series converge if and only if

$$\sum_{j=1}^{+\infty} y_{1-j} z_r^{-j} \sum_{k=1}^{r} a_k(x)(z_r/z_k)^j$$

or

$$\sum_{j=1}^{+\infty} y_j z_r^{-j} \sum_{k=1}^{r} a_k(1-x)(z_r/z_k)^j \tag{7}$$

converge, respectively. Now we consider the power series

$$\sum_{j=1}^{+\infty} y_{1-j} A_j(x)\, z^j, \qquad \sum_{j=1}^{+\infty} y_j A_j(1-x)\, z^j,$$

where, by (4) and definition,

$$0 < a_r(t) < A_j(t) := \sum_{k=1}^{r} a_k(t)(z_r/z_k)^j \leqslant \sum_{k=1}^{r} a_k(t) \leqslant A$$

is valid for some constant $A \in \mathbb{R}$ and for $0 < t < 1$.

Because of

$$\limsup_{j \to +\infty} \sqrt[j]{|y_{1-j}|\, A_j(x)} = \limsup_{j \to +\infty} \sqrt[j]{|y_{1-j}|},$$
$$\limsup_{j \to +\infty} \sqrt[j]{|y_j|\, A_j(1-x)} = \limsup_{j \to +\infty} \sqrt[j]{|y_j|},$$

their radii of convergence are R_+ and $1/R_-$, where

$$R_- := \limsup_{j \to \infty} \sqrt[j]{|y_{-j}|},$$

$$R_+ := 1/\limsup_{j \to \infty} \sqrt[j]{|y_j|},$$

respectively. Therefore the series of (7) converge if

$$|z_r^{-1}| < R_-^{-1} \qquad \text{or} \qquad |z_r^{-1}| < R_+,$$

respectively, and they diverge if

$$|z_r^{-1}| > R_-^{-1} \qquad \text{or} \qquad |z_r^{-1}| > R_+,$$

respectively. Hence it is convenient to introduce

$$R := \min\{R_+, R_-^{-1}\}$$

to be the "radius of convergence" of the "sequence" $\{y_j\}_{j \in \mathbb{Z}}$.

THEOREM. *Let R denote the radius of convergence of the sequence $\{y_j\}_{j \in \mathbb{Z}}$. The series*

$$s(x) = \sum_{j=-\infty}^{\infty} y_j l_j(x)$$

converges for $x \in \mathbb{Z}$ by its own. In case of $x \in \mathbb{R} \setminus \mathbb{Z}$, it converges if $R > |z_r^{-1}|$, and diverges if $R < |z_r^{-1}|$. If $R > |z_r^{-1}|$ holds, the series and their formal derivatives

$$s^{(k)}(x) = \sum_{j=-\infty}^{+\infty} y_j l_j^{(k)}(x), \qquad k = 0, 1, \ldots, m-1, \tag{8}$$

are uniformly convergent on any compact set in $\mathbb{R}$, and s is a solution of the interpolation problem

$$(S_m \ni s : s(k) = y_k \textit{ for } k \in \mathbb{Z}).$$

Proof. For $x = k \in \mathbb{Z}$ the series converges obviously and its value is y_k. The statements on the convergence or divergence are proved already if $0 < x < 1$. For $k-1 < x < k$, $k \in \mathbb{Z}$, they can be proved by shifting x and j. This does not affect the radius of convergence. Now let $R > |z_r^{-1}|$. If $0 \leqslant x \leqslant 1$, the series of (7) are dominated by the convergent series

$$\sum_{j=1}^{+\infty} |y_{1-j}|\,|z_r^{-1}| A$$

and

$$\sum_{j=1}^{+\infty} |y_j|\,|z_r^{-1}|^{j} A,$$

respectively. Hence the series of $s(x)$ converges uniformly on $[0, 1]$. By similar arguments, where $a_k(x)$ and $a_k(1-x)$ are replaced by the corresponding derivative, we obtain the result that all the formal derivatives of $s(x)$ up to the order $m-1$ converge uniformly on $[0, 1]$. For every other interval of the form $[k-1, k]$, $k \in \mathbb{Z}$, we obtain the same result by shifting methods. Hence all the series of (8) converge uniformly on every compact set in $\mathbb{R}$. From this it follows that $s \in S_m$, and this finishes the proof.

Remarks. The essential condition of the Theorem can also be read like

$$\begin{aligned} &\limsup_{j \to +\infty} \sqrt[j]{|y_j|} < |z_r|, \\ &\limsup_{j \to -\infty} \sqrt[|j|]{|y_j|} < |z_r|, \end{aligned} \tag{9}$$

hence it states for the y_j-s a growth not exceeding that of $|z_r|^{|j|}$ for $j \to \pm\infty$, which is exponential. Note again that $|z_r| > 1$.

Next let us introduce the (formal) Laurent-series

$$\sum_{j=-\infty}^{+\infty} y_j z^j.$$

Its upper and lower radii of convergence are R_+ and R_-, respectively. Because of $z_r^{-1} = z_{r+1}$, compare [5, formulas (6), (7)], e.g., we can also read the condition (9) like

$$|z_{r+1}| < R_+, \qquad R_- < |z_r|. \tag{10}$$

For the uniqueness of interpolating spline functions with exponential growth compare Greville *et al.* [1].

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